Solution to the Riemann-Hilbert boundary value problem for multiply connected domains

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1. Riemann-Hilbert problem for multiply connected domains

Consider mutually disjoint disks \( D_k = \{ z \in \mathbb{C} : |z - a_k| < r_k \}, \ k = 1, 2, \ldots, n \), in the complex plane \( \mathbb{C} \), \( D = \mathbb{C} \setminus \bigcup_{k=1}^{n} (D_k \cup \partial D_k) \).

Given Hölder continuous functions \( \lambda(t) \neq 0 \) and \( h(t) \). To find a function \( \phi(z) \) analytic in \( D \) continuous in \( D \cup \partial D \) with the following boundary condition

\[
\text{Re} \lambda(t) \phi(t) = h(t) \text{ on } |t - a_k| = r_k, \ k = 1, 2, \ldots, n.
\]

The problem (1.1) is a partial case of the \( \mathbb{R} \)-linear problem

\[
\phi^+(t) = a(t) \phi^-(t) + b(t) \overline{\phi^-(t)} + c(t) \text{ on } |t - a_k| = r_k, \ k = 1, 2, \ldots, n.
\]
History of the problem (RH): (sorry, not all is cited)

- Riemann (1851) Statement and discussion of the vector-matrix problem $\Phi^+ = G \Phi^-$ and a note about determination of analytic function by a relation between its Re and Im on the boundary.
- Poincaré (1882) Automorphic function and $\theta$-series.
- Volterra (1883) Statement and first simple results.
- Hilbert (1904, 1924) Reduction to singular integral equation with the kernel $\text{ctg} \frac{\sigma - s}{2}$.
- Muskhelishvili (1932), I. N. Vekua & Ruhadze (1933) (antiplane elastic problem, transmission problem, perfect contact)
- Golusin (1934) A functional equation for multiply connected domains and generalized alternating Schwarz method.
Gakhov (1941) Closed form solution for simply connected domains.

Kveselava (1945) Reduction to integral equations. Preliminary result on Bojarski's system.


I. N. Vekua (1952) Integral equations, estimation of the defect numbers.

Bojarski (1958-1959) Estimation of the defect numbers. Solvability of the problem had been reduced to solvability of the linear algebraic system with analytic coefficients (Bojarski's system). Special case $0 \leq \chi \leq n$.

Bojarski (1960) "elliptic case" of the R-linear problem (see L. G. Mikhajlov (1963)).


Mikhlin (1949) Generalized alternating Schwarz method (Decomposition methods) and Schwarz's operator.


Complete solution to the problem: Mityushev (1993-1998)

"Constructive methods to linear and non-linear boundary value problems for analytic function. Theory and applications", Chapman & Hall / CRC, 1999 (with S. V. Rogosin)
Auxiliary

$C_A$ - Banach space of functions analytic in $\bigcup_{k=1}^n D_k$ and continuous in $\bigcup_{k=1}^n (D_k \cup \partial D_k)$ with the norm $\| f \| = \max_{k=1,2,...,n} \max_{|t-a_k|=r_k} |f(t)|$. 
$z^*(k) := \frac{r_k^2}{z-a_k} + a_k$ - inversion with respect to the circle $|t-a_k| = r_k$.

If $f(z)$ is analytic in $|z - a_k| < r_k$, then $\overline{f(z^*(k))}$ is analytic in $|z - a_k| > r_k$.

Introduce $z^*_m(k_m k_{m-1} \ldots k_1) := (z^*_m(k_m-1 k_{m-2} \ldots k_1))^{*}_{k_m}$, which can be written in the form

$$\gamma_j(z) = \frac{e_j z + b_j}{c_j z + d_j}, \quad m \in 2 \mathbb{Z}, \quad \gamma_j(\bar{z}) = \frac{e_j \bar{z} + b_j}{c_j \bar{z} + d_j}, \quad m \in 2 \mathbb{Z} + 1, \quad e_j d_j - b_j c_j = 1, \quad m - \text{level of } \gamma_j$$

$$\gamma_0(z) = z, \quad \gamma_1(\bar{z}) = z^*_1, \quad \gamma_2(\bar{z}) = z^*_2, \ldots, \quad \gamma_n(\bar{z}) = z^*_n, \quad \gamma_{n+1}(z) = z^*_1, \quad \gamma_{n+2}(z) = z^*_2, \ldots$$

$$\mathcal{K} := \{\gamma_j(z) \text{ of even level}\}, \quad \mathcal{F} := \{\gamma_j(\bar{z}) \text{ of odd level}\}$$
2. Harmonic measures

Let \( w \in D \) be a fixed point. To find a function \( \alpha_s(z) = \alpha_s(x, y) \) (\( z = x + iy \)) harmonic in \( D \) continuous in \( D \cup \partial D \) with the following boundary condition

\[
\alpha_s(t) = \delta_{sl} \text{ on } |t - a_k| = r_k, \ k = 1, 2, \ldots, n,
\]

where \( \delta_{sl} \) is the Kronecker symbol. Represent the harmonic function \( \alpha_s(z) \) in the form

\[
\alpha_s(z) = \text{Re} \phi(z) + \sum_{m=1}^{n} A_m \ln |z - a_m| + A,
\]

where \( \sum_{m=1}^{n} A_m = 0, \ \phi(w) = 0. \)

1. \((2.1) \iff \text{R-linear problem: } \phi(t) = \phi_k(t) - \overline{\phi_k(t)} + f(t), \ |t - a_k| = r_k, \ k = 1, 2, \ldots, n, \)

where

\[
f(z) = \delta_{sl} - \sum_{m \neq k} A_m \ln |z - a_m| + A_k \ln r_k + A \ln |z - a_k| \leq r_k. \quad (2.3)
\]
Introduce

\[
\Phi(z) = \begin{cases} 
\phi_k(z) + \sum_{m \neq k} \left[ \phi_m(z^{*}(m)) - \phi_m(w^{*}(m)) \right] - \phi_k(w^{*}(k)) + f(z), & \mid z - a_k \mid \leq r_k, \\
\phi(z) + \sum_{m=1}^{n} \left[ \phi_m(z^{*}(m)) - \phi_m(w^{*}(m)) \right], & z \in D.
\end{cases}
\]

\(| t - a_k | = r_k: \quad \Phi^+ (t) - \Phi^- (t) = \phi(t) - \phi_k(t) + \phi_k(t) - f(t) = 0 \implies \Phi(z) = \phi(w) = 0 \]

2. $$\implies$$ system of **functional equations**

\[
\phi_k(z) = -\sum_{m \neq k} \left[ \phi_m(z^{*}(m)) - \phi_m(w^{*}(m)) \right] + \phi_k(w^{*}(k)) - f(z), \quad \mid z - a_k \mid \leq r_k, \quad k = 1, 2, \ldots, n,
\]

\[
\phi(z) = -\sum_{m=1}^{n} \left[ \phi_m(z^{*}(m)) - \phi_m(w^{*}(m)) \right], \quad z \in D.
\]
**Lemma.** Given \( f(z) \in C_A \) and numbers \( \mu_k \in [0, 2\pi) \). The system of functional equations

\[
\phi_k(z) = -e^{i\mu_k} \sum_{m \neq k} e^{-i\mu_m} \left[ \phi_m(z^*(m)) - \phi_m(w^*(m)) \right] + \phi_k(w^*(k)) - f(z), \quad (2.6)
\]

\[|z - a_k| \leq r_k, \quad k = 1, 2, \ldots, n,\]

has a unique solution in \( C_A \). This solution can be found by the method of successive approximations. The approximations converges in \( C_A \).

The proof of Lemma is based on solvability of the \( \mathbb{R} \)-linear problem Bojarski (1960)

Application of Lemma to (2.4) and (2.5) yields

\[
\phi(z) = -\sum_{m=1}^{n} A_m \left( \sum_{k \neq m} \ln \frac{z^*(k) - a_m}{w^*(k) - a_m} + \sum_{k \neq m, k_1 \neq k} \ln \frac{z^*(k_1) - a_m}{w^*(k_1) - a_m} + \ldots \right) = \\
\sum_{m=1}^{n} A_m \ln \prod_{j=1}^{\infty} \psi_m^j(z), \quad \text{where } \psi_m^j(z) = \begin{cases} 
\frac{\gamma_j(z) - a_m}{\gamma_j(w) - a_m} & \text{if } \gamma_j \in \mathcal{K}_m, \\
\frac{\gamma_j(z) - a_m}{\gamma_j(w) - a_m} & \text{if } \gamma_j \in \mathcal{F}_m,
\end{cases}
\]
\( \mathcal{K}_m \) (or \( \mathcal{F}_m \)) = \{ z^*(k_j,k_{j-1},...,k_1) \in \mathcal{K} : k_j \neq m \}.

Harmonic measures

\[
\alpha_s(z) = \sum_{m=1}^{n} A_m \ln | \prod_{j \in \mathcal{K}_m \cup \mathcal{F}_m} \psi_{m}^j(z) | + A_0.
\]

The real constants \( A_m \) (\( m = 0, 1, \ldots, n \)) satisfy a linear algebraic system. They are written explicitly.
3. Schwarz's operator

Complex Green's function is represented in the form

\[ M(z, \zeta) = M_0(z, \zeta) + \sum_{m=1}^{n} \alpha_m(\zeta) \ln(z - a_m) - \ln(\zeta - z) + A(\zeta), \quad (3.1) \]

where \( M_0(z, \zeta) \) is a single-valued analytic function of \( z \) in \( D \) (for any fixed \( \zeta \)), \( M_0(w, \zeta) = 0 \), \( A(\zeta) \) is unknown. The boundary value problem for \( M_0(z, \zeta) \)

\[ \text{Re}[M_0(t, \zeta) + \sum_{m=1}^{n} \alpha_m(z) \ln(t - a_m) - \ln(\zeta - t) + A(\zeta)] = 0, \quad |t - a_k| = r_k, \quad k = 1, 2, \ldots, n \quad (3.2) \]

is reduced to the system of functional equations

\[ \phi_k(z) = -\sum_{m \neq k} \left[ \phi_m(z^*(m)) - \phi_m(w^*(m)) \right] - f(z), \quad |z - a_k| \leq r_k, \quad k = 1, 2, \ldots, n. \quad (3.3) \]

Application of Lemma to (3.3) yields the formula
\[ M_0(z, \zeta) = \sum_{m=1}^{n} \alpha_m(z) \ln \prod_{j=1, j \neq m}^{\infty} \psi_m^j(z) + \omega(z, \zeta), \quad (3.4) \]

where \( \omega(z, \zeta) = \ln \prod_{j=1}^{\infty} \omega_j(z, \zeta), \quad \omega_j(z, \zeta) = \begin{cases} \frac{\zeta - \gamma_j(z)}{\zeta - \gamma_j(w)} & \text{if} \quad \gamma_j \in \mathcal{K}, \\ \frac{\zeta - \gamma_j(z)}{\zeta - \gamma_j(w)} & \text{if} \quad \gamma_j \in \mathcal{F}. \end{cases} \)

Schwarz's operator solves the problem \( \text{Re} \phi(t) = f(t) \) on \( |t - a_k| = r_k, \ k = 1, 2, \ldots, n \) with single-valued \( \text{Re} \phi(z) \) in \( D \) (\( \text{Im} \phi(z) \) is multi-valued in \( D \)). It is written explicitly in the following form

\[
\phi(z) = \frac{1}{2\pi i} \sum_{k=1}^{n} \int_{|\zeta - a_k| = r_k} f(\zeta) \times \left( \sum_{\gamma_j \in \mathcal{K}} \left( \frac{1}{\zeta - \gamma_j(w)} - \frac{1}{\zeta - \gamma_j(z)} \right) + \left( \frac{r_k}{\zeta - a_k} \right)^2 \sum_{\gamma_j \in \mathcal{F}} \left( \frac{1}{\zeta - \gamma_j(z)} - \frac{1}{\zeta - \gamma_j(w)} \right) - \frac{1}{\zeta - z} \right) d\zeta + \frac{1}{2\pi i} \sum_{k=1}^{n} \int_{|\zeta - a_k| = r_k} f(\zeta) \frac{\partial A}{\partial \nu}(\zeta) d\sigma + \sum_{k=1}^{n} A_k [\ln(z - a_k) + \psi_k(z)] + i\varnothing.
\]

It is possible to write explicitly \( A(\zeta) \).
4. Riemann-Hilbert problem

Using the factorization of $\lambda(t)$ reduce the problem (1.1) to

$$ \text{Re} e^{-i\mu_k} \omega(t) = g(t) \text{ on } |t - a_k| = r_k, \; k = 1, 2, ..., n. \quad (4.1') $$

**Remark.** It is possible to write explicitly $e^{-i\mu_k}$, $g(t)$ and the factorization function $X(z)$. The Riemann-Hilbert problem (4.1') is reduced to the functional equations

$$ \phi_k(z) = -e^{-i\mu_k} \sum_{m \neq k} e^{i\mu_m} [\phi_m(z^*(m)) - \phi_m(w^*(m))] + \phi_k(w^*(k)) + f(z), $$

$$ |z - a_k| \leq r_k, \; k = 1, 2, ..., n, \quad (4.2) $$

where $f(z)$ has the following structure $f(z) = h(z) + \sum_{s=1}^{2\chi} p_s \beta_s(z) + e^{-i\mu_k} Q$, $h(z)$ and $\beta_s(z)$ are known, $p_s$ are arbitrary real constants (for positive index $\chi$ of $\lambda(t)$), $Q$ is arbitrary complex constant. Note that $\beta_s(z)$ does not depend on $g(t)$ which enters into $h(z)$. 
The solution of (4.2) has the form

\[ \phi_k(z) = (\mathcal{A}_k h)(z) + \sum_{s=1}^{2^\chi} p_s(\mathcal{A}_k \beta_s)(z) + e^{-i\mu_k} Q + c, \]

where the operator \( \mathcal{A}_k \) is defined by the formula

\[
(\mathcal{A}_k F)(z) = e^{-i\mu_k} \sum_{m=0}^{\infty} (-1)^m \sum_{k_1 \neq k} \sum_{k_2 \neq k_1} \cdots \sum_{k_m \neq k_{m-1}} C^m e^{-i\mu_{k_m}} [F(z^*_{(k_m k_{m-1} \ldots k_1)}) - F(w^*_{(k_m k_{m-1} \ldots k_1)})] + F(z)
\]

\[ |z - a_k| \leq r_k, \]

\[ \omega(z) = Q - \sum_{m=1}^{n} e^{i\mu_k} [\phi_m(z^*_{(m)}) - \phi_m(w^*_{(m)})] \]  \hspace{1cm} (4.3)
Theorem 1 \( (\chi \geq 0) \)  Let at least two numbers \( \mu_k \) are non-equal. Then the Riemann-Hilbert problem (4.1) is solvable iff the linear algebraic system with respect to \( p_s \) and \( Q \) is solvable

\[
\text{Re}[- e^{-i\mu_k} \sum_{m \neq k} e^{i\mu_m} ([\mathcal{A}_k f] (w^*_m) - (\mathcal{A}_k f) (w^*_m))] -
\]

\[
\frac{\partial}{\partial \cdot} - \beta_{\mu_k} - \sum_{s=1}^{\chi} p_s \beta_s (w^*_s) + e^{-i\mu_k} Q = 0, \tag{4.4}
\]

\( k = 1, 2, \ldots, n. \)

If (2.6) is fulfilled, the general solution has the form

\[
\phi(z) = \prod_{m=1}^{n} (z - a_k)^{\chi_k} [X(z) \omega(z) + \sum_{s=1}^{\chi} \delta_z z^s]. \tag{4.5}
\]

In (4.5) \( \chi \) undetermined complex constants \( \delta_z \) are transformed into \( 2 \chi \) real \( p_s \).

Existence of the system (4.4) were predicted by Bojarski (see addition to I.N. Vekua  Generalized analytic functions, Nauka, Moscow, 1988).
**Theorem 2** \( (\chi < 0) \) Let at least two numbers \( \mu_k \) are non-equal. Then the Riemann-Hilbert problem (4.1) is solvable iff for some \( Q \)

\[
\text{Re}[-e^{-i\mu_k} \sum_{m \neq k} e^{i\mu_m} [(\mathcal{A}_k f)(w^*_m) - (\mathcal{A}_k f)(w^*_m)] - h(w^*_k) + e^{-i\mu_k} Q] = 0,
\]

\[
k = 1, 2, \ldots, n.
\]

If (2.6) is fulfilled, the general solution has the form

\[
\phi(z) = \prod_{m=1}^{n} (z - a_k)^{\chi_k} X(z) \omega(z).
\]

The solution \( \phi(z) \) is regular at infinity iff \( \omega(z) \) has zero at \( z = \infty \) of order \( \mid \chi \mid \).
Let $H(z)$ be a meromorphic function. The Poincaré $\theta_{2q}$–series

$$\theta_{2q}(z) := \sum_{\gamma_j \in \mathcal{K}} H[\gamma_j(z)] (c_j z + d_j)^{-2q}$$

is associated with the group $\mathcal{K}$.

When $q > 1$ the series (4.6) converges absolutely and almost uniformly in a subset of $\mathbb{C}$. When $q = 1$ the series (4.6) can be either absolutely convergent or absolutely divergent. Poincaré proposed to investigate the absolute convergence by comparison with the series $\sum_{j=0}^{\infty} |c_j|^{-2}$. Necessary and sufficient conditions for absolute convergence of the series have been found by [Akaza T., Inoue K. Limit sets of geometrically finite free Kleinian groups, Tohoku Math. J., 36, 1-16, 1984].

However, the series $\theta_2(z)$ is always uniformly convergent. More precisely [Mityushev V. Convergence of the Poincare series for classical Schottky groups, Proc. AMS, 126, 8, 2399-2406, 1998]
**Theorem 3.** Let a rational function $H(z)$ has poles only at regular points of $\mathcal{K}$. Then the Poincaré $\theta_2$-series converges uniformly in every compact subset of $D \setminus \text{limit points of } \mathcal{K}$.

Proof is based on the representation $\theta_2(z) = -\frac{1}{2} [(\phi(z) + H(z)) - (\omega(z) - H(z))]$, where $\phi(z)$ and $\omega(z)$ have the form of (2.5), where $\phi_k(z)$ satisfy similar functional equations.
5. Generalized method of Schwarz \( \text{Sch} \)

Dirichlet problem: \( u(t) = f(t) \) on \( |t - a_k| = r_k, \ k = 1, 2. \)

First approximation:

Second approximation:
Integral equations corresponding to the method

\[ u_k(z) = -\frac{1}{2\pi} \sum_{m \neq k} \int_{L_m} u_m(\tau) \left[ \frac{\partial G_m(z, \tau)}{\partial n} - \frac{\partial G_m(w, \tau)}{\partial n} \right] \, ds + \frac{1}{2\pi} \int_{L_k} u_k(\tau) \frac{\partial G_k(w, \tau)}{\partial n} \, ds + f(z), \]

\[ k = 1, 2, \ldots, n. \]

\( G_m(z, \tau) \) is Green's function of the exterior of \( D_m \).