Neutral coated inclusions of finite conductivity

Pawel Jarczyk and Vladimir Mityushev

Proc. R. Soc. A published online 16 November 2011

Advance online articles have been peer reviewed and accepted for publication but have not yet appeared in the paper journal (edited, typeset versions may be posted when available prior to final publication). Advance online articles are citable and establish publication priority; they are indexed by PubMed from initial publication. Citations to Advance online articles must include the digital object identifier (DOIs) and date of initial publication.

To subscribe to Proc. R. Soc. A go to: http://rspa.royalsocietypublishing.org/subscriptions

This journal is © 2011 The Royal Society
Neutral coated inclusions of finite conductivity

BY PAWEŁ JARCZYK AND VLADIMIR MITYUSHEV*

Department of Computer Sciences and Computer Methods, Pedagogical University, ul. Podchorąży 2, Kraków 30-084, Poland

We discuss the conductivity of two-dimensional media with coated neutral inclusions of finite conductivity. Such an inclusion, when inserted in a matrix, does not disturb the uniform external field. We are looking for shapes of the core and coating in terms of the conformal mapping $\omega(z)$ of the unit disc onto coated inclusions. The considered inverse problem is reduced to an eigenvalue problem for an integral equation containing singular integrals over a closed curve $L_1$ on the transformed complex plane. The conformal mapping $\omega(z)$ is constructed via eigenfunctions of the integral equation. For each fixed curve $L_1$, the boundary of the core is given by the curve $\omega(L_1)$. The boundary of the coating is obtained by the mapping of the unit circle. It is justified that any shaped inclusion with a smooth boundary can be made neutral by surrounding it with an appropriate coating. Shapes of the neutral inclusions are obtained in analytical form when $L_1$ is an ellipse.

Keywords: coated neutral inclusion; conformal mapping; boundary value problem

1. Introduction

Mathematical models of invisibility are of considerable interest in a number of recent publications. One can find the theoretical foundations and results devoted to various approaches of invisibility due to Kerker (1975), Ammari & Kang (2004, 2007), Alu & Engheta (2005), Milton & Nicorovici (2006), Milton et al. (2006), Farhat et al. (2008), Greenleaf et al. (2009), Guenneau et al. (2010), Liu (2010) and Ammari et al. (2011). In this paper, we investigate this problem in the context of conductivity of two-dimensional media with coated neutral inclusions. When the conductivity $\sigma_0$ of an isotropic matrix is chosen appropriately, one can insert a coated cylinder or sphere, with core conductivity $\sigma_1$ and coating conductivity $\sigma$, into the medium without disturbing the surrounding unidirectional external field. This effect is described in some detail by Milton (2000). Hashin (1962) constructed infinite packings of the plane by such coated cylinders of various sizes, also without disturbing the surrounding field. Hashin & Shtrikman (1962) established that the effective conductivity of such packings satisfy the famous Clausius–Mossoti approximation. These results were extended to coated ellipses and ellipsoids by Milton (1980, 1981). One can find an extended review with corresponding citations devoted to this problem in the book by Milton (2000) and in the paper by Milton & Serkov (2001).

*Author for correspondence (vmityushev@gmail.com).
Milton & Serkov (2001) have posed the question of whether there are neutral coated inclusions other than single and multi-coated circles, spheres, ellipses and ellipsoids. By having used the assumption that the coating surrounds a hole or a perfect conductor, Milton & Serkov (2001) solved the problem for two-dimensional geometry. They discovered many possible shapes of the neutral inclusions. This result was obtained by use of the conformal mapping of the doubly connected coated domain onto a circular annulus. As a result, a boundary value problem for analytic functions arises. The latter problem was solved by the use of Laurent’s series. It is worth noting that such problems were discussed by Schiffer (1959) and by Mityushev (1992) as well as Mityushev & Rogosin (1999) in the context of pure mathematical problems.

Our work is motivated by the question posed by Milton & Serkov (2001), which concerns a core of finite conductivity. Milton & Serkov (2001) noted that their analysis does not fit to this case, since they used a conformal mapping of the coating domain. Thus, the core domain cannot be taken into account apart from its boundary effects. Milton & Serkov (2001) tried to solve this problem approximately by small perturbations of the coated circles.

In this paper, the result of Milton & Serkov (2001) is extended to coated inclusions of finite conductivity represented in figure 1, where $G_1$, $G$, $G_2$ denote core, coating and the exterior domain; curve $G_1$ divides $G_1$ and $G$; curve $G_2$ divides $G$ and $G_2$. We also apply the method of conformal mappings, but, contrary to Milton & Serkov (2001), a conformal mapping $\zeta \mapsto z$ of the simply connected domain bounded by the exterior coated boundary $G_2$ onto the unit disc is used. This conformal mapping yields a boundary value problem that does not have an easier form than the original one. However, the unknown boundary $G_2$ transforms onto the known unit circle.

One of the main results of this paper is that the neutral inclusion problem is reduced to an eigenvalue $\mathbb{R}$-linear problem on the auxiliary plane $z$ (see §2b). In §3, the integral equation (3.12) corresponding to the eigenvalue problem is deduced. The solution of this problem depends on the curve $L_1$ dividing the transformed coated and core domains on the plane of variable $z$. It is worth noting that this eigenvalue problem has solutions for any Hölder continuous curve $L_1$. This
observation implies that any shaped inclusion with a smooth boundary can be
made neutral by surrounding it with an appropriate coating. Shapes of the neutral
inclusions are obtained in analytical form in §§4 and 5 when $L_1$ is an ellipse.

2. Eigenvalue $\mathbb{R}$-linear problem

In this section, the problem of neutral inclusion is stated as an inverse boundary
value problem for analytic functions with an unknown curve. This problem is
reduced to an eigenvalue problem for an integral equation.

(a) Statement of the problem and reduction to $\mathbb{R}$-linear problem

Let the plane of complex variable $\zeta = x + iy$ be divided onto domains $G_1, G, G_2$
(figure 1) by simple closed smooth curves $\Gamma_1$ and $\Gamma_2$ oriented in counter-clockwise
directions.

For definiteness, it is assumed that the origin belongs to $G_1$ and the domain
$G_2$ contains the infinite point. The domains $G_1$ and $G_2$ are simply connected, $G$
doubly connected; $G_1, G$ and $G_2$ are, respectively, the core, coated and exterior
domains occupied by materials with scalar conductivities $\sigma_1, \sigma$ and $\sigma_0$. The potentials $u_1(x, y), u(x, y)$ satisfy Laplace’s equation in $G_1, G$ and continuously
differentiable in the closures of these domains. It is assumed that the potential
$u_2(x, y)$ in the matrix $G_2$ is a linear function of the form

$$u_2(x, y) = -e_1 x - e_2 y,$$  \hspace{1cm} (2.1)

where $e_0 = (e_1, e_2)$ is the uniform field outside the coated inclusion. Since $u_1(x, y)$ satisfies Laplace’s equation in the simply connected domain $G_1$, it is represented
as the real part of a function analytic in $G_1$

$$u_1(x, y) = \frac{2\sigma}{\sigma + \sigma_1} \text{Re} \phi_1(\zeta), \quad \zeta \in G_1,$$  \hspace{1cm} (2.2)

where the coefficient in (2.2) is introduced for convenience. The potential in the
doubly connected domain $G$ is represented in the form

$$u(x, y) = \text{Re} \phi(\zeta) = \text{Re} [\tilde{\phi}(\zeta) + A \ln \zeta], \quad \zeta \in G,$$  \hspace{1cm} (2.3)

where the function $\tilde{\phi}(\zeta)$ is analytic and single-valued in $G$, and $A$ is a real
constant. It will be shown later that $A = 0$. Physically, this comes about because
there is no net charge in the inclusion. The potential (2.1) can be also written in
the complex form

$$u_2(x, y) = -\text{Re} \overline{e_0} \zeta,$$  \hspace{1cm} (2.4)

where the vector $e_0$ is represented as the complex number $e_0 = e_1 + ie_2$; the bar
denotes the complex conjugation.

Let $n$ be the unit outward normal vector to the curve $\Gamma_2$, and let $\partial/\partial n$ be the
outward normal derivative. Then the perfect contact between materials along the
curve $\Gamma_2$ is described by equations

$$u = u_2, \quad \sigma \frac{\partial u}{\partial n} = \sigma_0 \frac{\partial u_2}{\partial n} \quad \text{on } \Gamma_2.$$  \hspace{1cm} (2.5)
Along similar lines
\[ u = u_1, \quad \sigma \frac{\partial u}{\partial n} = \sigma_1 \frac{\partial u_1}{\partial n} \quad \text{on } \Gamma_1. \]  
\( (2.6) \)

Following Mityushev & Rogosin (1999), we transform the real conditions (2.5) and (2.6) into complex. Using (2.3) and (2.4), we rewrite the first relation (2.5) in the form
\[ \text{Re } \phi(\zeta) = -\text{Re } \frac{c_0}{e_0} \zeta, \quad \zeta \in \Gamma_2. \]  
\( (2.7) \)

Let \( s \) be the natural parameter of the curve \( \Gamma_2 \) and \( \sqrt{s} \) denote the corresponding derivative along \( \Gamma_2 \). Using the Cauchy–Riemann equation \( \sigma \frac{\partial v}{\partial s} = \sigma_0 \frac{\partial v_2}{\partial s} \), we rewrite the second equation (2.5) in the form
\[ \sigma v = \sigma_0 v_2 \quad \text{on } \Gamma_2. \]  
\( (2.8) \)

The integration constant in (2.8) is taken to be zero, since the imaginary part of the complex potential is defined up to an arbitrary additive constant. It follows from (2.4) that the imaginary part \( v_2(x, y) = -\text{Im } \frac{c_0}{e_0} \zeta \). Then (2.8) yields
\[ \sigma \text{Im } \phi(\zeta) = -\sigma_0 \text{Im } \frac{c_0}{e_0} \zeta, \quad \zeta \in \Gamma_2. \]  
\( (2.9) \)

The two real equations (2.7) and (2.9) can be written in the form of one complex relation:
\[ \phi(\zeta) = -\frac{\sigma_0 + \sigma}{2\sigma_0} \frac{c_0}{e_0} \zeta - \frac{\sigma_0 - \sigma}{2\sigma_0} \frac{c_0}{e_0} \bar{\zeta}, \quad \zeta \in \Gamma_2. \]  
\( (2.10) \)

The same arguments applied to (2.6) yield
\[ \phi(\zeta) = \phi_1(\zeta) - \sqrt{q} \tilde{\phi}(\zeta), \quad \zeta \in \Gamma_1, \]  
\( (2.11) \)

where the contrast Bergman parameter is introduced as follows:
\[ q = (\sigma_1 - \sigma)(\sigma_1 + \sigma)^{-1}. \]  
\( (2.12) \)

This parameter satisfies inequality \( |q| < 1 \) for finite positive \( \sigma_1 \) and \( \sigma \).

In order to prove that \( A = 0 \) in the representation (2.3), we calculate the increment \( [\phi]_{\Gamma_1} \) of the function \( \phi(\zeta) \) along the closed curve \( \Gamma_1 \). It follows from (2.3) that \( [\phi]_{\Gamma_1} = 2\pi i A \). On the other hand, \( [\phi]_{\Gamma_1} \) calculated by (2.11) yields zero. Hence, \( A = 0 \) in the representation (2.3) and \( \tilde{\phi}(\zeta) = \phi(\zeta) \) is a single valued function analytic in the doubly connected domain \( G \).

Thus, we arrive at the boundary value problem (2.10)–(2.11) with unknown curve \( \Gamma_2 \) with respect to the complex potentials \( \phi(\zeta), \phi_1(\zeta) \) analytic in \( G, \Gamma_1 \) and continuously differentiable in the closures of the domains considered.

\[ (b) \text{ Reduction to eigenvalue problem} \]

The simply connected domain \( G_1 \cup \Gamma_1 \cup G \) can be conformally mapped onto the unit disc \( U \). Let
\[ \zeta = -\frac{2\sigma_0}{(\sigma_0 - \sigma)e_0} \omega(z) \]  
\( (2.13) \)
denote the inverse conformal mapping of $U$ onto $G_1 \cup \Gamma_1 \cup G$. The coefficient in (2.13) is introduced for convenience. Let this conformal mapping be normalized by the condition $\omega(0) = 0$. The unit circle $L_2$ transforms, as the boundary of $U$, onto the curve $\Gamma_2$, a simple closed curve $L_1 \subset U$ onto $\Gamma_1$ (figure 1).

The domains $D_1$ and $D$ are transformed by (2.13) onto $G_1$ and $G$, respectively, the point $z = 0$ belongs to $D_1$. Introduce the complex potentials analytic in the considered domains $D_1$ and $D$

$$\varphi_1(z) = \varphi_1(\zeta) \quad \text{and} \quad \varphi(z) = \phi(\zeta),$$

where the variables $z$ and $\zeta$ are related by (2.13). The conjugation conditions (2.10) and (2.11) become

$$\varphi(t) = \omega(t) - \lambda \omega(t), \quad |t| = 1$$

and

$$\varphi(t) = \varphi_1(t) - g \varphi_1(t), \quad t \in L_1,$$

where

$$\lambda = (\sigma + \sigma_0)(\sigma - \sigma_0)^{-1}.$$ (2.17)

One can see that $\lambda^{-1}$ is equal to the second contrast parameter and $|\lambda| > 1$.

It is convenient to introduce the function $\varphi_2(z) := \omega(z^{-1})$ for $|z| \geq 1$ analytic in $|z| > 1$ and continuously differentiable in $|z| \geq 1$. The function $\varphi_2(z)$ vanishes at infinity, since $\omega(0) = 0$. Then (2.15)–(2.16) takes the form of the standard $\mathbb{R}$-linear condition (Mityushev & Rogosin 1999)

$$\varphi(t) = \varphi_2(t) - \lambda \varphi_2(t), \quad |t| = 1$$

and

$$\varphi(t) = \varphi_1(t) - g \varphi_1(t), \quad t \in L_1.$$ (2.19)

It is necessary to find a curve $L_1$ for which the problem (2.18)–(2.19) has a non-trivial solution. Then the non-zero function

$$\omega(z) = \varphi_2(z^{-1}), \quad |z| \leq 1$$

can be a candidate for the conformal mapping. The function (2.20) besides being analytic must be a one-to-one map of $U$ onto $G_1 \cup \Gamma_1 \cup G$.

Following Milton & Serkov (2001), it is more convenient to treat the above problem as an eigenvalue problem, with respect to the unknown constant $\lambda$ and a fixed curve $L_1$ as follows: to find non-trivial functions $\varphi(z)$, $\varphi_1(z)$ and $\varphi_2(z)$ analytic in $D$, $D_1$ and $D_2$, respectively, and continuously differentiable in the closures of the domains considered with the $\mathbb{R}$-linear conjugation conditions (2.18) and (2.19). The constant $\lambda$ has to be determined. Moreover, $\varphi_2(z)$ vanishes at infinity. The contrast parameter $g$ is supposed to be fixed in such a way that $|g| < 1$.

3. Integral equation

In the present section, the $\mathbb{R}$-linear problem (2.18) and (2.19) is reduced to an integral equation. Let $D_2$ be the exterior of the unit circle. Let the curve $L_1$ and the unit circle $L_2$ be oriented in the counter-clockwise direction. Then the boundaries
of the open domains $D_1$, $D$ and $D_2$ are expressed by the relations $\partial D_1 = L_1$, $\partial D_2 = -L_2$ and $\partial D = L_2 \cup (-L_1)$. Let $\Psi^+(z)$ and $\Psi^-(z)$ be functions analytic in the domains $D$ and $D^- = D_1 \cup D_2$, respectively, and Hölder continuous in their closures. Then, Cauchy’s integral formula implies the following relations described by Gakhov (1966):

\[
\frac{1}{2\pi i} \int_{\partial D} \frac{\Psi^+(t)}{t - z} \, dt = \begin{cases} 
\Psi^+(z) & \text{for } z \in D \\
0 & \text{for } z \in D^-
\end{cases}
\] (3.1a)

\[
\frac{1}{2\pi i} \int_{L_2} \frac{\Psi^-(t)}{t - z} \, dt = \begin{cases} 
\Psi^-(\infty) - \Psi^-(z) & \text{for } |z| > 1 \\
\Psi^-(\infty) & \text{for } |z| < 1
\end{cases}
\] (3.1b)

and

\[
\frac{1}{2\pi i} \int_{L_1} \frac{\Psi^-(t)}{t - z} \, dt = \begin{cases} 
\Psi^-(z) & \text{for } z \in D_1 \\
0 & \text{for } z \in D \cup L_1 \cup D_2
\end{cases}
\] (3.1c)

Let the function $\mu(t)$ be Hölder continuous on $\partial D$. The Cauchy-type integral

\[
\Psi(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{\mu(t)}{t - z} \, dt
\] (3.2)

represents a function analytic in the domains $D$, $D^-$ and it satisfies the jump condition (Gakhov 1966)

\[
\Psi^+(t) - \Psi^-(t) = \mu(t), \quad t \in \partial D,
\] (3.3)

where $\Psi^\pm(z)$ can be considered as the restriction of $\Psi(z)$ to $D \cup \partial D$ and to $D^- \cup \partial D$ or as the limit values of $\Psi(z)$ from the different sides of the curves $\partial D$. One can also consider formula (3.2) as the unique solution to the jump problem (3.3) with $\Psi^-(\infty) = 0$ due to Gakhov (1966).

The relations (2.18) and (2.19) with

\[
\Psi^+(z) = \varphi(z), \quad z \in D;
\Psi^-(z) = \begin{cases} 
\varphi_1(z) & \text{if } z \in D_1 \\
\varphi_2(z) & \text{if } |z| > 1
\end{cases}
\] (3.4)

can be written in the form of the jump condition (3.3) with

\[
\mu(t) = \begin{cases} 
-\lambda \varphi_2(t) & \text{if } t \in L_2 \\
-\varphi_1(t) & \text{if } t \in L_1
\end{cases}
\] (3.5)

Then (3.2) becomes the system of integral equations

\[
\varphi_1(z) = -\frac{\lambda}{2\pi i} \int_{|t|=1} \frac{\varphi_2(t)}{t - z} \, dt + \frac{\theta}{2\pi i} \int_{L_1} \frac{\varphi_1(t)}{t - z} \, dt, \quad z \in D_1, \quad (3.6)
\]

and

\[
\varphi_2(z) = -\frac{\lambda}{2\pi i} \int_{|t|=1} \frac{\varphi_2(t)}{t - z} \, dt + \frac{\theta}{2\pi i} \int_{L_1} \frac{\varphi_1(t)}{t - z} \, dt, \quad |z| > 1. \quad (3.7)
\]
The relation $t = \bar{t}^{-1}$ is fulfilled in the unit circle. The function $\varphi_2(\bar{t}^{-1})$ is analytically continued into the unit disc, since $\varphi_2(z)$ is analytic in $|z| > 1$. Then the first integral in (3.6) can be calculated by Cauchy’s formula:

$$\frac{1}{2\pi i} \int_{|t|=1} \frac{\varphi_2(\bar{t}^{-1})}{t - z} \, dt = \begin{cases} \varphi_2 \left( \frac{1}{z} \right) & \text{if } z \in D_1 \\ 0 & \text{if } |z| > 1. \end{cases}$$ (3.8)

Substitution of (3.8) into (3.6) and (3.7) yields

$$\varphi_1(z) = -\lambda \varphi_2 \left( \frac{1}{z} \right) + \frac{\varrho}{2\pi i} \int_{L_1} \frac{\varphi_1(t)}{t - z} \, dt, \quad z \in D_1$$ (3.9)

and

$$\varphi_2(z) = \frac{\varrho}{2\pi i} \int_{L_1} \frac{\varphi_1(t)}{t - z} \, dt, \quad |z| > 1.$$ (3.10)

Substitute $\bar{z}^{-1}$ instead of $z$ in (3.10) and take the complex conjugation

$$\varphi_2 \left( \frac{1}{\bar{z}} \right) = -\frac{\varrho}{2\pi i} \int_{L_1} \frac{\varphi_1(t)}{t - (1/z)} \, d\bar{t}, \quad |z| < 1.$$ (3.11)

The function $\varphi_2(z)$ can be eliminated from the system (3.6)–(3.7), which is reduced to the integral equation

$$\varphi_1(z) = \varrho \left\{ \frac{\lambda}{2\pi i} \int_{L_1} \frac{\varphi_1(t)}{t - (1/z)} \, d\bar{t} + \frac{1}{2\pi i} \int_{L_1} \frac{\varphi_1(t)}{t - z} \, dt \right\}, \quad z \in D_1.$$ (3.12)

This integral equation has to be stated as the following eigenvalue problem. To find a non-zero function $\varphi_1(z)$ and a constant $\lambda$ which satisfies equation (3.12). The function $\varphi_1(z)$ has to be analytic in $D_1$ and Hölder continuous in $D_1 \cup L_1$.

Similar problems were already discussed by Bojarski (1960) and by Schiffer (1959), who considered the eigenvalue problem (3.6)–(3.7) when $q = \lambda$. Schiffer (1959) proved that the eigenfunctions of that problem generate a complete orthogonal basis in a Hilbert space; all eigenvalues are real, generate a countable set and their moduli are greater than unity. This result does not fit to the considered case with fixed $q$ satisfying the inequality $|q| < 1$. However, it is possible to extend Schiffer’s result to the problem (3.6)–(3.7) or to the equivalent problem (3.12). This result will be presented in a separate paper with necessary mathematical description of the functional space and singular integral operators.

In order to study the properties of the coatings and its dependence on the core, one can numerically solve the integral equation (3.12) for various $L_1$ and $\varrho$. Applying polynomial approximations for the function $\varphi_1(z)$, consider some examples of a numerical solution to (3.12). All the computations are performed for the normalized conductivity of the matrix $\sigma_0 = 1$. Let the shape of the core of conductivity $\sigma_1 = 3.36$ be given by the curve $\zeta = 0.6e^{i\theta} + 0.31e^{2i\theta} + 0.07e^{3i\theta} + 0.013e^{4i\theta} \ (0 \leq \theta < 2\pi)$. Three neutral coatings with different conductivity $\sigma$ are presented in figure 2. It is interesting to note that the non-convex of the core have convex and non-convex exterior boundaries of the coatings.
Another example with the boundary curve of core $\zeta = 0.5e^{i\theta} + 0.35e^{2i\theta} + 0.12e^{3i\theta} + 0.06e^{4i\theta}$ ($0 \leq \theta < 2\pi$) is presented in figure 3. Here, we are looking for similar shapes of the coatings for different $\sigma_1$. All the found shapes differ locally near the point $z = -0.5$, where the boundary of coatings approaches to the core with increasing $\sigma_1$.

4. Functional equation

As noted at the end of §3, it is convenient to consider (3.12) as an eigenvalue problem with a fixed contour $L_1$. Each fixed contour $L_1$ on the plane $z$ produces a neutral inclusion on the plane $\zeta$. In order to constructively solve the integral equation (3.12), one can take an algebraic curve $L_1$ and reduce (3.12) to a functional equation. Any smooth curve can be approximated by an algebraic one. The simplest algebraic curve $L_1 = \mathbb{v}D_1$ is a circle. One can check that the circle $L_1$ yields the circular annulus on the physical plane $z$ described by Milton (2000). In the present section, $D_1$ is taken as an ellipse. This gives other non-trivial shapes of the neutral coated domain.

Let semi-axis of the ellipse $L_1$ be denoted by $r(1 + \alpha)$ and $r(1 - \alpha)$ ($0 < \alpha < 1$, $r > 0$). The Joukowsky conformal mapping

$$z(w) = r \left( w + \frac{\alpha}{w} \right)$$

(4.1)

transforms the annulus $\sqrt{\alpha} < |w| < 1$ onto $D_1 \setminus \Gamma$, where $\Gamma$ is the slit $(-2\sqrt{\alpha}, 2\sqrt{\alpha})$ along the $x$-axis (figure 4). The mapping (4.1) transforms $D'$ onto $D$ and $D'_2$ onto $D_2$. The curve $L'_2$ bounds a Bell’s domain (Bell et al. 2009). The inverse
Figure 3. Closed neutral shapes of coated inclusions: (a) whole picture; (b) fragment near the point $z = -0.5$. For various conductivity of the core $\sigma_1$ conductivity of coating takes the corresponding $\sigma$: $\sigma_1 = 1.58, \sigma = 172$ (solid line), $\sigma_1 = 2.79, \sigma = 1.50$ (dashed line) and $\sigma_1 = 29.9, \sigma = 1.18$ (dotted line).

mapping to (4.1) has the form

$$w(z) = \frac{1}{2} \left( \frac{z}{r} + \sqrt{\frac{z^2}{r^2} - 4\alpha} \right), \quad (4.2)$$

where the branch of the root is chosen in such a way that $\lim_{z \to X \pm 0} \sqrt{\frac{z^2}{r^2} - 4\alpha} = \pm i\sqrt{4\alpha - X^2}$, for $-2\sqrt{\alpha}r < X < 2\sqrt{\alpha}r$.

Let $\tau$ run over the unit circle. Then $t = r(\tau + \alpha/\tau)$ runs over $L_1$. Introduce the function

$$\Phi_1(w) = \varphi_1 \left( w + \frac{\alpha}{w} \right) = \varphi_1(z), \quad (4.3)$$

analytic in $\sqrt{\alpha} < |w| < 1$, ($z \in D_1 \setminus T$) and Hölder continuous in $\sqrt{\alpha} \leq |w| \leq 1$. It satisfies the condition

$$\Phi_1(w) = \Phi_1 \left( \frac{\alpha}{w} \right), \quad |w| = \sqrt{\alpha}. \quad (4.4)$$
Figure 4. Transformation of the domains $D_1$ and $D_2$ on the plane $z$ onto $D_1'$ and $D_2'$ on the plane $w$. The domain $(D_2')^*$ is obtained from $D_2'$ by the inversion $w \rightarrow 1/\bar{w}$ with respect to the unit circle.

Equation (4.4) implies that the function $\Phi_1(w)$ can be written in the form

$$\Phi_1(w) = \Phi(w) + \Phi\left(\frac{\alpha}{w}\right), \quad \sqrt{\alpha} \leq |w| \leq 1,$$

where $\Phi(w)$ is analytic in $|w| < 1$.

We now proceed to transform equations (3.9) and (3.10) on the plane $z$ to deduce functional equations on the plane $w$. Similar method can be applied to (3.12) with the same result.

Let $z$ belong to $D_1$. The integral from (3.9) becomes

$$\frac{1}{2\pi i} \int_{L_1} \frac{\varphi_1(t)}{t - z} \, dt = \frac{1}{2\pi i} \int_{|\tau| = 1} \frac{[\Phi(\tau) + \Phi(\alpha/\tau)](1 - \alpha/\tau^2) \, d\tau}{\tau + (\alpha/\tau) - w - (\alpha/w)},$$

where $z = \tau(w + \alpha/w), |w| < 1$. The integral (4.6) can be calculated by residues. First, consider the integral

$$J_1 = \frac{1}{2\pi i} \int_{|\tau| = 1} \frac{\Phi(1/\bar{\tau})(\tau^2 - \alpha) \, d\tau}{\tau(\tau - w)(\tau - (\alpha/w))},$$

where the function $\Phi(1/\bar{\tau})$ is analytically continued into $|\tau| > 1$. It follows from inequality $\sqrt{\alpha} < |w| < 1$ that all roots of the denominator of (4.7) belong to the unit disc except at infinity. Hence,

$$J_1 = \frac{1}{2\pi i} \int_{|\tau| = 1} \frac{\Phi(1/\bar{\tau})(\tau^2 - \alpha) \, d\tau}{\tau(\tau - w)(\tau - (\alpha/w))} = \text{res}_{\tau=\infty} = \overline{\Phi(0)}.$$

The second part of the integral (4.6) can be calculated by residues at the points $\tau = 0; w; \alpha/w$ lying in the unit disc:

$$J_2 = \frac{1}{2\pi i} \int_{|\tau| = 1} \frac{\Phi(\alpha\bar{\tau})(\tau^2 - \alpha) \, d\tau}{\tau(\tau - w)(\tau - (\alpha/w))} = -\overline{\Phi(0)} + \Phi(\alpha/w) + \Phi\left(\frac{\alpha^2}{w}\right).$$
Neutral coated inclusions

since the function \( \Phi(\alpha \tau) \) is analytic in \(|\tau| < 1\). The integral (4.6) is equal to \( J_1 + J_2 \). Substitution of this result into (3.9) and the use of (4.3) and (4.5) yields

\[
\Phi(w) + \Phi\left(\frac{\alpha}{w}\right) = -\varphi_2 \left(\frac{1}{\bar{z}}\right) + \varphi \left[ \Phi(\alpha \bar{w}) + \Phi\left(\frac{\alpha^2}{w}\right) \right],
\]

(4.10)

where \( \sqrt{\alpha} \leq |w| \leq 1 \).

Now we transform the integral (3.10) written in the form

\[
\frac{1}{2\pi i} \int_{|\tau|=1} \frac{\overline{\Phi(1/\bar{\tau})}(\tau^2 - \alpha) d\tau}{\tau (\tau - w)(\tau - (\alpha/w))},
\]

(4.11)

where \( z = r(w + \alpha/w) \) with \(|z| > 1\), which implies that \(|w| > 1\). Calculate the first part of the integral (4.11) by residues in \(|w| > 1\)

\[
\frac{1}{2\pi i} \int_{|\tau|=1} \frac{\overline{\Phi(1/\bar{\tau})}(\tau^2 - \alpha) d\tau}{\tau (\tau - w)(\tau - (\alpha/w))} = \text{res}_{\tau=w} + \text{res}_{\tau=\infty} = -\Phi\left(\frac{1}{w}\right) + \Phi(0). \tag{4.12}
\]

The second part of the integral (4.11) is calculated by residues in \(|w| < 1\)

\[
\frac{1}{2\pi i} \int_{|\tau|=1} \frac{\overline{\Phi(\alpha \bar{\tau})}(\tau^2 - \alpha) d\tau}{\tau (\tau - w)(\tau - (\alpha/w))} = \text{res}_{\tau=0} + \text{res}_{\tau=\alpha/w} = -\Phi(0) + \Phi\left(\frac{\alpha^2}{w}\right). \tag{4.13}
\]

Substitution of (4.12) and (4.13) into (4.11) yields

\[
\varphi_2(z) = \varphi \left[ \Phi\left(\frac{\alpha^2}{w}\right) - \Phi\left(\frac{1}{w}\right) \right] \tag{4.14}
\]

for \( z \) and \( w \) related by (4.1) and (4.2) when \(|z| > 1\) and \( w \in D' \). Introduce the function \( \Phi_2(w) \) analytic in \( D'_2 \)

\[
\Phi_2(w) = \varphi_2 \left[ r \left( w + \frac{\alpha}{w} \right) \right]. \tag{4.15}
\]

The relation (4.15) can be written on the plane \( z \)

\[
\varphi_2(z) = \Phi_2 \left( \frac{1}{2} \left( \frac{z}{r} + \sqrt{\frac{z^2}{r^2} - 4\alpha} \right) \right), \quad |z| \geq 1. \tag{4.16}
\]

It follows from (4.16) that \( \varphi_2(1/\bar{z}) = \Phi_2(\beta(\bar{w})) \), where

\[
\beta(\bar{w}) = \frac{r + \sqrt{r^2 - 4\alpha(\bar{w} + (\alpha/\bar{w})^2)}}{2(\bar{w} + (\alpha/\bar{w}))}.
\]
Therefore, equations (3.9) and (3.10) take the form
\[
\Phi(w) + \Phi\left(\frac{\alpha}{w}\right) = -\lambda \Phi_2[\bar{\beta}(\bar{w})] + \varrho \left[ \Phi(\alpha \bar{w}) + \Phi\left(\frac{\alpha^2}{w}\right) \right], \quad \sqrt{\alpha} \leq |w| \leq 1,
\]
and
\[
\Phi_2(w) = \varrho \left[ \Phi\left(\frac{\alpha^2}{w}\right) - \Phi\left(\frac{1}{\bar{w}}\right) \right], \quad |w| > 1.
\] (4.17)

Substitute \( \Phi_2 \) from the second equation of (4.17) to the first one
\[
\Phi(w) + \Phi\left(\frac{\alpha}{w}\right) = -\lambda \varrho \{ \Phi[\alpha^2 f(w)] - \Phi[f(w)] \} + \varrho \left[ \Phi(\alpha \bar{w}) + \Phi\left(\frac{\alpha^2}{w}\right) \right],
\] (4.18)
where
\[
f(w) = [\bar{\beta}(\bar{w})]^{-1} = \frac{1}{2\alpha} \sum_{n=1}^N \frac{(2n)!r^{2(n-1)}\alpha^n}{(2n-1)(n)!^2} \left( w + \frac{\alpha}{w} \right)^{2n-1} + O(r^{4N}).
\] (4.19)

This functional equation (4.18) can also be obtained from equation (3.12).

Introduce the operator \( P^+ \), which transforms Laurent’s series to its regular part. More precisely, let \( h(w) = \sum_{k=-\infty}^{+\infty} h_k w^k, \sqrt{\alpha} < |w| < 1. \) Then \( (P^+ h)(w) = \sum_{k=0}^{\infty} h_k w^k, \) \( |w| < 1. \) Application of \( P^+ \) to equation (4.18) yields
\[
\Phi(w) = \varrho \Phi(\alpha \bar{w}) - \lambda \varrho P^+ \{ \Phi[\alpha^2 f(w)] - \Phi[f(w)] \}, \quad |w| \leq 1.
\] (4.20)

Thus, we arrive at the following eigenvalue problem. To find \( \Phi(w) \) analytic in \(|w| < 1\) and continuous in \(|w| \leq 1\) with a constant \( \lambda \) satisfying equation (4.20).

\section{5. Solution to functional equation}

In this section, we solve the functional equation (4.20) using the polynomial approximation
\[
\Phi(w) \approx \sum_{m=1}^M \alpha_m w^m
\] (5.1)
with a fixed number \( M \). The operator from the right-hand part of (4.20) is compact in the Banach space endowed with the norm \( ||\Phi|| = \max_{|w| \leq 1} |\Phi(w)| \) (see Mityushev 1984). Therefore, approximate solutions of (4.20) can be found by replacement of (4.20) with equations of finite rank operators. Substitution of (5.1) into (4.20) and selection of the coefficients on \( w^m \) \( (m = 0, 1, \ldots, M) \) yields such an equation. Having used these arguments, Mityushev (1984) showed that the polynomial from (5.1) tends to \( \Phi(w) \) as \( M \) tends to infinity.

We also use approximations on \( r^2 \) \( (r < 1) \). More precisely, all symbolic computations are performed with the accuracy \( O(r^{2(2N-1)}) \). For definiteness, the power \( 2(2N-1) \) is taken with an odd number \( 2N - 1 \). Substitute (5.1) and (4.19) in the functional equation (4.20), and then select the terms with
the same powers $w^m$ for $m = 0, 1, \ldots, M$. As a result, we obtain a system of homogeneous linear algebraic equations with respect to the coefficients $\alpha_m$ and the spectral parameter $\lambda$. This eigenvalue problem can be solved by a standard scheme. Substitution of the rational approximation from (4.19) into (4.20) instead of $f(w)$ is justified by the compactness of the operator containing $f(w)$. In order to illustrate the algorithm, we take $M = 4$ and $N = 8$. Then the system becomes

\[
\begin{align*}
- \alpha_1 + r^2\lambda \alpha_1 + \alpha q \alpha_1 &- r^2 \lambda \alpha^2 q \alpha_1 + 3r^6 \lambda \alpha^2 \alpha q \alpha_1 - 3r^6 \lambda \alpha^4 \alpha q \alpha_1 \\
+ 3r^6 \lambda \alpha q \alpha_3 - 3r^6 \lambda \alpha^7 \alpha q \alpha_3 &- 0 \\
- \alpha_2 + r^4 \lambda \alpha_2 + \alpha^2 q \alpha_2 &+ 8r^8 \lambda \alpha^2 \alpha q \alpha_2 - r^4 \lambda \alpha^4 \alpha q \alpha_2 + 8r^8 \lambda \alpha^6 \alpha q \alpha_2 \\
+ 4r^8 \lambda \alpha q \alpha_4 - 4r^8 \lambda \alpha^9 \alpha q \alpha_4 &- 0 \\
r^6 \lambda \alpha \alpha_1 - r^6 \lambda \alpha^3 \alpha q \alpha_1 &- \alpha_3 + r^6 \lambda \alpha_3 + \alpha^3 q \alpha \alpha_3 - r^6 \lambda \alpha^6 \alpha \alpha_3 = 0 \\
2r^8 \lambda \alpha q \alpha_2 - 2r^8 \lambda \alpha^3 \alpha q \alpha_2 &- \alpha_4 + r^8 \lambda \alpha \alpha_4 + \alpha^4 q \alpha \alpha_4 - r^8 \lambda \alpha^5 \alpha q \alpha_4 = 0.
\end{align*}
\]

(5.2)

Denote $\lambda_j$, $(j = 1, \ldots, 4)$ the eigenvalues of the system (5.2). For brevity, only the first eigenvalue $\lambda$ is explicitly written

\[
\lambda_1 = [2r^8(-1 + \alpha^2)^2(1 + \alpha^2 + \alpha^4)q^{-1}(r^2(-1 + \alpha^2)q(-1 + \alpha^3 q + r^4 \\
\times (-1 + \alpha(q + \alpha(-4 + \alpha(q + \alpha(-1 + 4\alpha q))))))) - \sqrt{(r^4(-1 + \alpha^2)^2 q^2 \\
\times ((-1 + \alpha^3 q)^2 + 2r^4(-1 + \alpha^2)(-1 + \alpha^3 q)(-1 + \alpha(q + \alpha(1 + 4\alpha q))))}(2)))).
\]

(5.3)

Each $\lambda_j$ yields a non-zero solution of the system (5.2). If $\lambda = \lambda_1$, then $\alpha_1$ is an arbitrary number, $\alpha_3 = k_1 \alpha_1$ and $\alpha_2 = \alpha_4 = 0$, where

\[
k_1 = r^4 \alpha(-1 + \alpha q + \alpha^3 q - \alpha^4 q^2 - r^4(1 - \alpha^2))(1 - \alpha^3 q)^{-2} \\
\times [(1 - 2\alpha q - \alpha^3 q + 2\alpha^4 q^2 - \alpha^2(1 - \alpha^2))].
\]

(5.4)

If $\lambda = \lambda_2$, then $\alpha_2$ is arbitrary, $\alpha_3 = k_2 \alpha_1$ and $\alpha_1 = \alpha_3 = 0$. If $\lambda = \lambda_3$, then $\alpha_3$ is arbitrary, $\alpha_3 = k_3 \alpha_1$ and $\alpha_2 = \alpha_4 = 0$. If $\lambda = \lambda_4$, then $\alpha_4$ is arbitrary, $\alpha_2 = k_4 \alpha_1$ and $\alpha_1 = \alpha_3 = 0$. Coefficient $k_j$ tends to zero as $r \to 0$. This means that $|k_j|$ is small for sufficiently small $r$. We do not write them all, because of their long form. The normalized eigenvectors corresponding to the eigenvalues (5.3) have the form

\[
\begin{align*}
\Phi^{(1)}(w) &\approx w + k_1 w^3, \quad \Phi^{(2)}(w) \approx w^2 + k_2 w^3, \\
\Phi^{(3)}(w) &\approx k_3 w + w^3 \quad \text{and} \quad \Phi^{(4)}(w) \approx k_4 w + w^4.
\end{align*}
\]

(5.5)

The map $\omega : U \to G_1 \cup \Gamma_1 \cup G$ is a one-to-one if and only if $\varphi_2(z)$ is a one-to-one map. The function $\varphi_2(z)$ is related to $\Phi(w)$ by (4.14). Therefore, $\varphi_2(z)$ is one-to-one in $|z| > 1$ if and only if the function $F(w) = \Phi(\varphi^2 w) - \Phi(w)$ is one-to-one in the domain $(D^*_p)^*$, the image of $D^*_p$ under the inversion with respect to unit circle (figure 4). The function $\Phi(w)$ is one for the functions (5.5). Consider the first function $\Phi^{(1)}(w)$. Then $F(w) = -w - k_1 w^3 + w \alpha + k_1 w^3 \alpha^3$. Let $w_0$ belong to
Table 1. Structure of the eigenvalues. Empty places correspond to zero.

<table>
<thead>
<tr>
<th></th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$\alpha_3$</th>
<th>$\alpha_4$</th>
<th>$\alpha_5$</th>
<th>$\alpha_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1$</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda_3$</td>
<td></td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda_4$</td>
<td></td>
<td></td>
<td></td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda_5$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$\lambda_6$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>

$(D_2^*) \subset U$. Then equation $F(w_0) = F(w)$ with respect to $w_0$ has the following three roots:

$$w_1 = w_0, \quad w_{2,3} = -\frac{w_0}{2} \pm \frac{\sqrt{-(4 + 3k_1 w_0^2(1 + \alpha^2 + \alpha^4))}}{2\sqrt{k_1(1 + \alpha^2 + \alpha^4)}}.$$  \hspace{1cm} (5.6)

For sufficiently small $r$, the roots $w_2$ and $w_3$ lie outside the unit disc. This implies that $F(w)$ is one-to-one mapping in $(D_2^*)^*$. Similar calculations can be done for $F(w)$ with $\Phi^{(j)}(w)$ ($j = 2, 3, 4$) given by (5.5). In all these cases, $F(w)$ is not a one-to-one mapping.

In the case $M = 6$ and $N = 12$, we have six values of $\lambda$ for which a system similar to (5.2) has non-zero solutions. The result is summarized in table 1.

For instance, $\alpha_1 = 1$, $\alpha_3 = k_{11}$, $\alpha_4 = k_{12}$, $\alpha_2 = \alpha_4 = \alpha_6 = 0$ if $\lambda = \lambda_1$. The coefficients $k_{ij}$ are not written here because of their long form. We only note that $|k_{ij}|$ are sufficiently small for small $r$. This implies that the eigenfunction $\Phi^{(1)}(w) \approx w + k_{11} w^3 + k_{12} w^5$ is a one-to-one mapping.

For arbitrary even $M$ and $N$ ($N \geq 2M$), we have $M$ eigenvalues and eigenfunctions. Only the first function realizes one-to-one mapping. It can be approximately represented in the following form:

$$\Phi(w) \approx w + k_{N1} w^3 + \cdots + k_{N,M/2} w^M,$$  \hspace{1cm} (5.7)

where $k_{N,j}$ tends to zero as $r \to 0$. The conformal mapping $\omega : U \to G_1 \cup \Gamma_1 \cup G$ is constructed via $\Phi$ by using (2.20) and (4.14):

$$\omega(z) = q \left[ \Phi \left( \frac{\alpha^2}{w(z)} \right) - \Phi \left( \frac{1}{w(z)} \right) \right],$$  \hspace{1cm} (5.8)

where $w(z)$ is given by (4.2). Consider an example based on the approximation (5.5). Substitution of (5.5) into (5.8) yields $\omega(z) \approx z q(1 + r^2 z^2 \alpha)(-1 + \alpha^2)$. Examples of computations are presented in figure 5.

6. Conclusion

The coated neutral inclusion problem for two-dimensional media is formulated as the eigenvalue $\mathbb{R}$-linear problem (2.18)–(2.19) on the auxiliary complex plane $z$. The later problem is reduced to the integral equation (3.12), where the closed curve $L_1$ can be arbitrarily fixed in the unit disc.
Neutral coated inclusions

Each solution of the discussed eigenvalue problem can produce a solution for the neutral coated problem, if only the function $\omega(z)$ from (2.20) is a one-to-one map. Examples given in §5 show that the following conjecture can be posed. There exists at least one eigenfunction of the problem (3.9)–(3.10) or of (3.12) so that the corresponding function $\omega(z) = \varphi_2(1/\overline{z})$ constructed by (3.11) is a one-to-one map of the unit disc onto $G_1 \cup \Gamma_1 \cup G_2$. This conjecture is equivalent to the existence of at least one eigenfunction from an infinite set of all eigenfunctions such that $\omega(z)$ in $|z| < 1$ has the unique zero at origin. Integral equation (3.12) gives a constructive algorithm to solve the neutral inclusion problem, since eigenvalues $\lambda$ and eigenfunctions $\phi_1(z)$ can be computed by standard methods due to Krasnosel’skij et al. (1969). Then $\omega(z)$ is constructed by (2.20) and (3.11). Furthermore, zeros of $\omega(z)$ in the unit disc have to be investigated (see for instance Kravanja & Van Brel 2000). If $z = 0$ is the unique zero of $\omega(z)$ in the unit disc, the shape of the neutral inclusion is given by formula $\zeta = \omega(e^{i\theta})$, $0 \leq \theta \leq 2\pi$.

In the earlier mentioned algorithm, $\varrho$ is a fixed number and $\lambda$ is unknown. Using formulae (2.12) and (2.17), it is easy to state the eigenvalue problem in which conductivity $\sigma$ has to be found. The spectral parameter $\lambda$ is a function of $\varrho$, i.e. $\lambda = \lambda(\varrho)$ (see for instance (5.3) when $L_1$ is an ellipse). Then $\sigma$ can be determined from equation $(\sigma + \sigma_0)/(\sigma - \sigma_0) = \lambda((\sigma_1 - \sigma)/(\sigma_1 + \sigma))$ with fixed $\sigma_0$ and $\sigma_1$.

It is difficult to precisely compare our results with Milton & Serkov (2001) since the $\mathbb{R}$-linear problem (2.18)–(2.19) does not coincide with the corresponding problem (3.17)–(3.18) from Milton & Serkov (2001) in the limit cases when $\sigma_1 = 0$ or $\sigma_1 = +\infty$. Moreover, it is difficult to match two results taken from two different infinite sets of all results. We can suggest that fig. 2(a) and (b) from Milton & Serkov (2001) can be obtained by using our approach when $L_1$ is an ellipse.

The results obtained in this paper allow us to make the following conclusion. Any two-dimensional core, i.e. a core of an arbitrary smooth shape and of an arbitrary conductivity, can be coated by such a material that the coated inclusion inserted in a matrix of an arbitrary fixed conductivity does not disturb the uniform field outside the inclusion. This conclusion can be justified by the following arguments. Any simply connected domain $G_1 \cup \Gamma \cup G$ can be conformally mapped onto the unit disc; $L_1$ in figure 4 is the image of $\Gamma_1$ in figure 1. For any Hölder continuous curve $L_1$, the integral equation (3.12) has non-trivial eigenfunctions. For sufficiently small ratios of the areas $|G_1|/(|G_1| + |G|)$ (that

![Figure 5. Neutral shapes of coated inclusions. (a) $\alpha = 0.2$, $\varrho = 0.9$ and $r = 0.8$, (b) $\alpha = 0.7$, $\varrho = 0.8$ and $r = 0.5$.](image-url)
is equivalent to small $r$ in §§4 and 5), one of the eigenfunctions produces the required conformal mapping $\omega(z)$. Therefore, this eigenfunction determines the shape of the coating $\Gamma_2$ (figure 1), and the corresponding eigenvalue determines the conductivity $\sigma$ of the coating. It is not yet known whether the area of coating $|G|$ can be small, i.e. $|G_1|/(|G_1| + |G|)$ is of order 1. This question is related to the condition that $\omega(z)$ must be a one-to-one map. The earlier mentioned discussion is restricted to inclusions with smooth boundaries. The physical and geometrical properties of the coating inclusions can be systematically investigated via the integral equation (3.12). Some examples are discussed at the end of §3.

References


